**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part I. Variations calculus

## Lecture 9. Variational problems with pointwise constraints

We have the method of the analysis for the problem of the minimization of the integral functional, which depends from one or many unknown functions of one or many variables. The functional can depend from derivatives of unknown functions of arbitrary order. We considered the problems with additional constraints on the boundary of the given set or isoperimetric conditions. The last case can be analyzed by means of Lagrange multipliers method. However there exist a lot of practical problems with pointwise state constraints for the value of unknown functions at the interior points of the functions domain. We will extend by means of Lagrange multipliers method to this class of the variational problems. The oscillation of the pendulum problem will be considered as an example.

**9.1. Problem statement**

Consider the integral



where *F* is a given function, and unknown functions  …,  satisfy boundary conditions

 (9.1)

with given values  and ,  Besides we have the additional conditions

 (9.2)

where the functions  are given,  Suppose the functions *F* and  are smooth enough.

**Problem 9.1.** *Find the vector-function v*, *which minimize the integral I and satisfies the equalities* (9.1), (9.2).

**9.2. Lagrange multipliers method**

We will use the standard variational technique. Determine the function



where *σ* is the number,  is smooth vector-function on the interval , satisfying homogeneous boundary conditions

 (9.3)

and additional equalities

 (9.4)

The set of admissible values *h* is non-empty because we have *n* equality (9.4) with *m* functions  and 

The integral *I* has the minimum in the point *u* if and only if the function *f* has the number 0 as the point of the minimum.

**Lemma 9.1**. *The derivative of the function f at the point zero is equal to*

 (9.5)

*where* 

**Proof**. Determine



Using the Taylor’s formula, we find



where  as  Then we get



After devising by *σ* and passing to the limit as  we get



Integrate by the parts with using the boundary conditions (9.1); we obtain



Then the previous equality is transformed to the formula (9.5).

Unfortunately, we cannot to use the Basic Lemma of the Calculus Variations for transformation the equality

 (9.6)

because *h* is not arbitrary here; is satisfies the equality (9.4). Therefore, we will transform this equality. Using Taylor’s formula we have



where  is the derivative of the function  with respect to  and  as  Use (9.4) after devising by *σ* and passing to the limit; we get

 (9.7)

We have the system of *n* linear algebraic equations with *m* unknown  Therefore,  functions  only are arbitrary in the formulas (9.6). Other functions can be determined by the equalities (9.7). Suppose  are arbitrary functions.

Multiply the equalities (9.7) by arbitrary functions ; after addition by *j* and integration by *x* we get



Add the equality with (9.6); we obtain



Determine the function

****  (9.8)

where  The previous equality can be transform to

 (9.9)

Chose the functions  such that the following relations are true.

 (9.10)

It can be transformed to the equalities



We have *n* linear algebraic equations with respect to unknown functions .

Then the equality (9.9) can be transformed to

 (9.11)

The functions  are arbitrary here. Choose *hk* equal to zero for all  We have



So we obtain

 (9.12)

Using equalities (9.10) and (9.12) we have the following result.

**Theorem 9.1.** *The solution of the Problem* 9.1 *satisfies the system of Euler equations*

 (9.13)

Hence our necessary conditions of extremum includes *m* second order differential equations (9.13) with boundary conditions

 (9.14)

and *n* algebraic equations

 (9.15)

for finding  and *Lagrange multipliers* .

The algorithm of solving of Problem 9.1 is given in Table 9.1.

Table 9.1. The algorithm of solving of Problem 9.1.

|  |  |
| --- | --- |
| **step** | **Action** |
| 1 | Determine the functions *F* and *Fj* and the constants   for the concrete Problem 9.1. |
| 2 | Determine the function *H* by the formula (9.8). |
| 3 | Determine Euler equations (9.13). |
| 4 | Find the general solutions of Euler equations;  it depends from 2*m* arbitrary constants and *n* Lagrange multipliers. |
| 5 | Put the general solution to the system (9.14), (9.15). |
| 6 | Find 2*m* arbitrary constants and *n* Lagrange multipliers  from the system (9.14), (9.15). |
| 7 | Put the values  to the formula (8.10) with the concrete function Ф; the result is the required functions. |
| 8 | Calculate the value of the corresponding value of the integral *I*. |

**9.3. Example**

Consider the problem of the minimization the integral



on the set of the functions   which satisfy boundary conditions

 (9.16)

and additional equality

 (9.17)

Determine the function



Then we have the system (9.13):

, .

After addition we have



Using (9.17), we obtain



then,  So we get Euler equations

 

We have two identical equations with different boundary conditions (9.16). The general solution of these both equations is the sum of the general solution of the corresponding homogeneous equation and partial solution of the initial equation. We chose the last value as a constant. It is equal to  So we find general solutions

 

with four arbitrary constants. Using (9.15) we find   So we have the equalities

 

with two arbitrary constants. By (9.17) we have



so  Hence the system of the conditions of the extremum has the solution

 

with arbitrary constant *c*.

Find the derivatives

 .

Put these values to the integral



So we have the same value of the integral for all solution of the necessary conditions of the minimum. It can be the solutions of our problem.

**9.4. Oscillation of the pendulum**

Consider the phenomenon of the oscillation of the pendulum. The state of the pendulum is characterized by the horizontal and vertical coordinates *x* and *y*. Let zero is the origin of the coordinates. Find the energy of the pendulum



The kinetic energy is determine by the vector of the velocity . Then



where *m* is the mass of the pendulum. The potential energy is determined by the weight



where *g* is gravitational acceleration. Then the energy of the oscillation of the pendulum is



The quantity of the energy on the time interval from  to  is



We know also that the distance between the pendulum and the origin of the coordinates is a constant *l*. Therefore, we have the additional condition

 (9.18)

By the minimum action principle the law of the movement of the pendulum satisfies the minimum of the value *I* with condition (9.18).

Using Theorem 9 we determine the function



We obtain Euler equations (9.13)

 (9.19)

 (9.20)

Hence we have the system (9.18) – (9.20) for finding three unknown values *х*, *у* and *λ*.

We transform Descartes coordinates to the polar coordinates with using the formulas



The equalities (9.19), (9.20) are transformed to



Multiply first equality by  and second one multiply by ; devise second equality by first one, we get

 (9.21)

where  The equality (9.20) is the *equation of the pendulum oscillation*, where the constant *ω* is called the *frequency of the oscillation*. If our oscillation is small enough, then  and equation of the pendulum oscillation has the standard form



Hence we find the mathematical model of the analyzed phenomenon.

**Outcome**

* The minimization problem with pointwise conditions can be solved by means of Lagrange multipliers method.
* The quantity of the Lagrange multipliers is equal to the quantity of the additional conditions.
* The necessary conditions of minimum for the problem with pointwise condition consist of second order differential Euler equations with two given boundary conditions and pointwise conditions with respect to unknown functions and Lagrange multipliers.
* The oscillation of the pendulum is in application of this theory.

### Task. Variational problem with pointwise condition

Find the functions   that minimize the integral



on the set of the functions with boundary conditions



and additional condition

. (\*)

The values of the parameters.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| variant | *L* | *X* | *Y* | *α* | *a* | *b* |
| 1 | 1 | 1 | 1 | -π2 | π | -π |
| 2 | π | 1 | 2 | -1 | -1 | 1/2 |
| 3 | π/2 | 1 | 1 | -2π2 | π | -π |
| 4 | π/2 | 2 | 2 | -2 | -2 | 2 |
| 5 | 4 | 2 | 2 | -4π2 | -π | π |
| 6 | π | 1 | 1 | -1 | 2 | -2 |
| 7 | 2 | 2 | 1 | -π2 | π/2 | -π |
| 8 | π | 2 | 2 | -1 | -1 | 1 |
| 9 | 4 | π | π | -π2 | -π | π/2 |
| 10 | 1 | 1 | 1 | -1 | 1 | -1 |
| 11 | π | 1 | 1 | -π2 | π | -π |
| 12 | 2 | 1 | 1 | -1 | -1 | 1 |
| 13 | π | 1 | 2 | -π2 | -π | π/2 |
| 14 | π | 2 | 1 | -1/2 | 1/2 | -1 |
| 15 | π | 2 | 2 | -2 | -2 | 2 |
| 16 | 2 | 2 | 2 | -π2 | π/2 | -π |
| 17 | π | 2 | 1 | -1 | π/2 | -π |
| 18 | π | 1 | 2 | -π2 | 2π | -π |

Steps of the task:

1. Denote the concrete problem statement.
2. Denote the system of the extremum conditions (concrete Euler equations with boundary and addition conditions).
3. Multiply the first Euler equation by the given parameter *a*, and second equation by *b*. Add these equalities with using of the condition (\*). Find the value *λ*.
4. Put *λ* to Euler equations.
5. Find the general solutions of two Euler equations. It equals the sum of the general solution of the corresponding homogeneous equation and the constant, which satisfies the given equation.
6. Find four constants from general solutions of Euler equations with using of the boundary conditions.
7. Put these constants to the formulas of the general solutions. It will be the solution of the problem.
8. Check the feasibility of the given constraint for it.

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### Next step

We considered the problems of functionals minimization for the case of the direct dependence of functionals from unknown functions. However, there exist many problems with non-direct form of this dependence. The unknown functions can be interpreted here as the controls. The minimizing functional is depended from state functions for these problems. Besides state functions satisfy state equations and depend from controls. These problems are called the optimization control problems. Thus, we will have the final part this course. It will be optimization methods.